THE STABILITY OF THE ROTATIONAL MOTIONS OF A SOLID BODY WITH A LIQUID CAVITY

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The author here considers the stability of the rotation of a solid body with a cavity filled entirely or partially (containing a bubble) with an ideal imcompressible homogeneous liquid.

Zhukovskii gave the general solution of the problem on the motion of a solid body with cavities filled completely with an imcompressible liquid [1]. On the basis of Zhukovskii's results, Chetaev [2] solved the precisely stated problem on the stability of the rotational motion of a solid body with cavities filled completely with an ideal liquid in vortexless motion.

Along with the exact methods, there were developed approximate procedures [3] for solving the problem on the motion of a solid body with liquid filled cavities.

Sobolev [4] applied the methods of functional analysis to the linearly formulated problem on the stability of the rotational motion of a body with cavities filled completely with an ideal liquid. The work of Krein [5] is devoted to the application of Sobolev's method to the solution of the problem on the motion of a body with liquid filled cavities.

On the other hand, the theory of the motion of a body with cavities that are not completely filled with a liquid has as yet not been sufficiently developed. Researches along this line deal mainly with the problem of small oscillations around the equilibrium position of the vessel with the liquid. This topic was considered in the works of G.E. Pavlenko, L.N. Stretenskii, D.E. Okhotsimskii, B.I. Rabinovich, N.N. Moiseev, and G.S. Narimanov.

Of special importance in research problems on the stability of solid bodies are certain works on the study of the stability of the forms of relative equilibrium of a rotating liquid, in particular, the works of Liapunov [6,7] and Poincaré [8]. In his work [8], Poincaré studied the small oscillations of a free liquid subject to Newton's law of gravitation. He considered the oscillations of the liquid in the neighborhood of its relative equilibrium position, and obtained certain results on the stability of figures of equilibrium. Poincaré confined his considerations to the linear equations of the first approximation, which he obtained by neglecting in the differential equations of the problem certain terms that he assumed to be small. Since the validity of the replacements of the original equations of the problem by linear equations cannot be proved, and since such a replacement simply substitutes for the original problem a new one whose solution may have nothing in common with the original one, it was pointed out by Liapunov [7] that the results obtained by Poincaré cannot be considered as having been rigorously established.

Liapunov revealed the difficulties which one meets in the study of the stability of continuous media; thus, in particular, there can occur in a liquid extensions which are large in linear dimensions but small in volume, are threadlike or laminar and contain little energy. In this connection it was shown by Liapunov that one is not justified in drawing conclusions on the basis of analogies between the case of a finite number of degrees of freedom and the case of motion of continuous media.

As is well known, the general problem on the stability of motion of continuous media has not been formulated up to the present time; the attempts made in the literature to state this problem as a problem on the stability with a finite number of degrees of freedom cannot be considered to have been successful.

In the present work there is chosen a different line of approach whose principal idea can be described as follows.

In the problems on the stability of motion of liquid-filled bodies, we are interested chiefly in the question of the stability of motion of the solid body; the question of the stability of motion of the liquid is of interest to us only in so far as the motion of the liquid exerts an effect on the stability of motion of the body as a whole (it is of course obvious that these two aspects of the problem are interconnected). In this connection it is natural to put the question of the stability of the motion of our system relative to all the variables which characterize the motion of the solid body and the motion of the liquid. In such a setting the problem of the stability of motion of the solid body, and of the liquid contained in its interior, leads to the investigation of the conditional stability of the system, that is, the stability relative to certain ones of the variables, and not to all of them that determine the motion of the mechanical system with an infinite number of variables. This problem is solved in this paper with the aid of Liapunov's second method and by starting out with the complete equations of the perturbed motion.

1. Equations of the perturbed motion of the system. Let us suppose that the central ellipsoid of inertia of the solid body is an ellipsoid of revolution, and that the completely or partially filled cavity of the body has the form of a solid of revolution. The liquid is assumed to be an ideal one. The center of inertia of the system will be taken as the origin of the system of coordinates, $O_1x_1y_1z_1$, whose axes will have fixed directions in space. The equations of motion will be referred to a system of coordinates, $Oxyz$, which is fixed with respect to the body, and whose origin O coincides with the center of inertia of the body, while the axes are directed along the principal axes of inertia of the body. Let the axis of rotation of the cavity coincide with the axis of rotation of the ellipsoid of inertia of the body; we take this axis to be the Oz-axis.

We shall consider the case when the equations of motion of the system relative to the center of inertia [2] can be written in the form [9]

$$
\begin{gathered}
A\frac{d\omega_1}{dt}+\frac{dg_1}{dt}+(C-A)\,\omega_2\omega_3+\omega_2g_3-\omega_3g_2=a\gamma_2\\
A\frac{d\omega_2}{dt}+\frac{dg_2}{dt}+(A-C)\,\omega_3\omega_1+\omega_3g_1-\omega_1g_3=-a\gamma_1\\
C\frac{d\omega_3}{dt}+\frac{dg_3}{dt}+\omega_1g_2-\omega_2g_1=0
\end{gathered}
\tag{1.1}
$$

$$
\begin{aligned}
&\frac{d}{dt}(u+v_1+\omega_2z-\omega_3y)+\omega_2(w+v_3+\omega_1y-\omega_2x)-\\
&\qquad-\omega_3(v+v_2+\omega_3x-\omega_1z)=-\frac{1}{\rho}\frac{\partial p}{\partial x}\\
&\frac{d}{dt}(v+v_2+\omega_3x-\omega_1z)+\omega_3(u+v_1+\omega_2z-\omega_3y)-\\
&\qquad-\omega_1(w+v_3+\omega_1y-\omega_2x)=-\frac{1}{\rho}\frac{\partial p}{\partial y}\\
&\frac{d}{dt}(w+v_3+\omega_1y-\omega_2x)+\omega_1(v+v_2+\omega_3x-\omega_1z)-\\
&\qquad-\omega_2(u+v_1+\omega_2z-\omega_3y)=-\frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}
\tag{1.2}
$$

$$
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.3}
$$

$$
\frac{d\gamma_1}{dt}=\omega_3\gamma_2-\omega_2\gamma_3,\qquad \frac{d\gamma_2}{dt}=\omega_1\gamma_3-\omega_3\gamma_1,\qquad \frac{d\gamma_3}{dt}=\omega_2\gamma_1-\omega_1\gamma_2 \tag{1.4}
$$

Here $A = B$, C denote the principal central moments of inertia of the body, a is a constant which characterizes the moment of the tilting couple; γ_1, γ_2 and γ_3 are direction cosines of the axis of the fixed direction O_1z_1, relative to the axes x, y and z of the moving system of coordinates; ω_1, ω_2, ω_3 are the projections on the axes x, y, z of the vector of the instantaneous angular velocity of the body, g_1, g_2, g_3 are the projections on the x, y and z axes of the vector of the moment of momentum of the liquid in its motion relative to the system of coordinates

$Ox_1y_1z_1$; v_1, v_2, v_3 are the projections on the x, y and z axes of the vector of the velocity of the center of inertia O of the body in its motion relative to the inertia O_1 of the system; u, v and w are the projections of the vector of the velocity of the particles of the liquid relative to the solid body in its motion relative to the system of coordinates $O_1x_1y_1z_1$; ρ is the density of the liquid, p is the pressure.

It is not difficult to see [9] that the equations of motion (1.1) to (1.4) of the system possess the following first integrals:

$$T_1 + T_2 + a\gamma_3 = h, \qquad (A\omega_1 + g_1)\gamma_1 + (A\omega_2 + g_2)\gamma_2 + (C\omega_3 + g_3)\gamma_3 = k$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \qquad \omega_3 = \omega_3^\circ \tag{1.5}$$

where T_1 stands for the kinetic energy of the body, T_2 is the kinetic energy of the liquid, $u = a\gamma_0$ is the potential energy of the external forces acting on the system, h, k and ω_3^0 are constants of integration.

It is easily seen that the equations (1.1) to (1.4) have the following particular solution:

$$\omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega, \qquad v_1 = v_2 = v_3 = 0$$
$$g_1 = g_2 = 0, \qquad g_3 = g, \qquad \gamma_1 = \gamma_2 = 0, \qquad \gamma_3 = 1 \tag{1.6}$$

which corresponds to the uniform rotation of the solid body around the Oz-axis, which here is parallel to the axis O_1z_1, while the motion of the liquid is stabilized and the liquid, in particular, can be (under certain conditions) in the state of relative equilibrium

$$u = v = w = 0, \qquad g = \omega\rho \int_{\tau_0} (x^2 + y^2)\, d\tau \tag{1.7}$$

or may be at rest relative to the system $O_1x_1y_1z_1$:

$$u = \omega y, \qquad v = -\omega x, \qquad w = 0, \qquad g = 0 \tag{1.8}$$

(for example, in the case of the vortexless motion of the liquid [1]). Here τ denotes the region of the space xyz occupied by the liquid at the given moment, and τ_0 stands for the region τ in the unperturbed motion.

The motion of the system described by the particular solution (1.6) will be taken as the non-perturbed motion of the body and of the liquid within the cavity. We shall investigate the stability of this motion.

Equation (1.1) shows that the motion of a solid body depends on the kinetic moment of the liquid and on its rate of change which, in turn, depends on the motion of the body. In this connection it is natural to investigate the stability of motion of our system relative to the projections ω_1, ω_2, ω_3 of the instantaneous angular velocity of the body, and relative to the projections g_1, g_2, g_3 of kinetic moment of the liquid upon the fixed axes, and also relative to certain parameters which characterize the position of the body in space. Such parameters can be, for example, the Euler angles, of the Krylov angles; below we select for such parameters the direction cosines γ_1, γ_2, γ_3 of the $O_1 z_1$-axis relative to the x, y, z-axes. These parameters are chosen as quantities which directly characterize the tilting moment that acts on the system. We note that the kinetic moment of the liquid determines as an integral (but not completely) the motion of the liquid. From this it follows that the stability of motion of the liquid relative to the projection (upon the moving axes) of the kinetic moment of the liquid is a conditional stability, that is, a stability of the motion of the liquid relative to some but not to all of the variables which characterize the motion.

Thus, we shall study the stability, in the sense of Liapunov, of the rotational motions of the body and liquid relative to the variables

$$\omega_1,\ \omega_2,\ \omega_3,\ g_1,\ g_2,\ g_3,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ v_1,\ v_2,\ v_3 \qquad (1.9)$$

which for the nonperturbed motion take on the given constant values (1.6).

In this manner we have reduced the problem of the stability of motion of the considered system with an infinite number of degrees of freedom to the investigation of the stability of the system relative to a finite number of quantities (1.9).

Let us construct the equations of the perturbed motions of the systems which will be applicable to the case when the perturbed motion is a uniform rotation of the body and liquid as one solid body. For the case when the unperturbed motion of the liquid is the state of rest, one needs only to set $g = 0$ in the results. If for the perturbed motion we set

$$\omega_3 = \omega + \xi, \qquad g_3 = g + \eta, \qquad \gamma_3 = 1 + \zeta \qquad (1.10)$$

and substitute (1.10) into the equations (1.1) to (1.4), we obtain the following equations for the perturbed motion of the system:

$$A\frac{d\omega_1}{dt} + \frac{dg_1}{dt} + (C - A)(\omega + \xi)\,\omega_2 + (g + \eta)\,\omega_2 - g_2(\omega + \xi) = a\gamma_2$$

$$A\frac{d\omega_2}{dt} + \frac{dg_2}{dt} + (A - C)(\omega + \xi)\,\omega_1 + g_1(\omega + \xi) - (g + \eta)\,\omega_1 = -a\gamma_1 \qquad (1.11)$$

$$C\frac{d\xi}{dt} + \frac{d\eta}{dt} + g_2\omega_1 - g_1\omega_2 = 0$$

$$\frac{du}{dt} + \frac{dv_1}{dt} + \frac{d\omega_2}{dt} z - \frac{d\xi}{dt} y + 2[\omega_2 w - (\omega + \xi) v] + \omega_2 v_3 - (\omega + \xi) v_2 +$$
$$+ \omega_1[(\omega_2 y + (\omega + \xi) z] - [\omega_2^2 + (\omega + \xi)^2] x = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{dv}{dt} + \frac{dv_2}{dt} + \frac{d\xi}{dt} x - \frac{d\omega_1}{dt} z + 2[(\omega + \xi) u - \omega_1 w] + (\omega + \xi) v_1 - \omega_1 v_3 +$$
$$+ \omega_2[(\omega + \xi) z + \omega_1 x] - [(\omega + \xi)^2 + \omega_1^2] y = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (1.12)$$
$$\frac{dw}{dt} + \frac{dv_3}{dt} + \frac{d\omega_1}{dt} y - \frac{d\omega_2}{dt} x + 2(\omega_1 v - \omega_2 u) + \omega_1 v_2 - \omega_2 v_1 +$$
$$+ (\omega + \xi)(\omega_1 x + \omega_2 y) - (\omega_1^2 + \omega_2^2) z = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
$$\frac{d\gamma_1}{dt} = (\omega + \xi)\gamma_2 - \omega_2(1 + \zeta), \qquad \frac{d\gamma_2}{dt} = \omega_1(1 + \zeta) - (\omega + \xi)\gamma_1$$
$$\frac{d\zeta}{dt} = \omega_2\gamma_1 - \omega_1\gamma_2 \qquad (1.13)$$

where

$$g_1 = \rho\int_\tau [y(w + v_3 + \omega_1 y - \omega_2 x) - z(v + v_2 + (\omega + \xi) x - \omega_1 z)]\, d\tau \qquad (1.14)$$
$$g_2 = \rho\int_\tau [z(u + v_1 + \omega_2 z - (\omega + \xi) y) - x(w + v_3 + \omega_1 y - \omega_2 x)]\, d\tau$$
$$\eta = \rho\int_\tau [x(v + v_2 + (\omega + \xi) x - \omega_1 z) - y(u + v_1 + \omega_2 z - (\omega + \xi) y)]\, d\tau -$$
$$- \omega\rho\int_{\tau_0} (x^2 + y^2)\, d\tau$$

If the cavity has the shape of a body of revolution and if $A = B$, then, as is shown in the work [9], there exists the integral

$$\omega_3 = \omega_3^\circ = \text{const}$$

and hence, during the entire course of motion,

$$\xi = \omega_3^\circ - \omega = \text{const} \qquad (1.15)$$

It is possible to establish three more integrals for the equations (1.11) to (1.13).

Let us multiply equations (1.11) by ω_1, ω_2, ω_3, respectively, and add the results; let us multiply equation (1.12) by u, v, w, respectively and add them. We multiply the result by $\rho\, d\tau$ and integrate over the total volume τ occupied by the liquid, and then we add this result to the equation obtained earlier. In consequence of this we will have an equation from which we obtain directly the following integral

$$\frac{1}{2}M(v_1^2+v_2^2+v_3^2)+\frac{1}{2}(A\omega_1^2+A\omega_2^2+2C\omega\xi+C\xi^2)+$$
$$+\frac{1}{2}\rho\int_\tau(u^2+v^2+w^2)\,d\tau+\frac{1}{2}\rho\int_\tau\{[\omega_2 z-(\omega+\xi)y]^2+[(\omega+\xi)x-\omega_1 z]^2+$$
$$+(\omega_1 y-\omega_2 x)^2\}\,d\tau+\rho\int_\tau\{u[v_1+\omega_2 z-(\omega+\xi)y]+v[v_2+(\omega+\xi)x-\omega_1 z]+$$
$$+w(v_3+\omega_1 y-\omega_2 x)\}\,d\tau+\rho\int_\tau\{v_1[\omega_2 z-(\omega+\xi)y]+v_2[(\omega+\xi)x-\omega_1 z]+$$
$$+v_3(\omega_1 y-\omega_2 x)\}\,d\tau+a\zeta=\text{const} \tag{1.16}$$

Here $M = M_1 + M_2$ is the mass of the system. This integral can be represented, in view of (1.14), in the following form

$$\frac{1}{2}M(v_1^2+v_2^2+v_3^2)+\frac{1}{2}[A\omega_1^2+A\omega_2^2+2C\omega\xi+C\xi^2]+$$
$$+\frac{1}{2}\rho\int_\tau(u^2+v^2+w^2)\,d\tau+\frac{1}{2}[\omega_1 g_1+\omega_2 g_2+\omega\eta+(g+\eta)\xi]+ \tag{1.17}$$
$$+\frac{1}{2}\rho\int_\tau\{(u+v_1)[\omega_2 z-(\omega+\xi)y]+(v+v_2)[(\omega+\xi)x-\omega_1 z]+$$
$$+(w+v_3)(\omega_1 y-\omega_2 x)\}\,d\tau+\rho\int_\tau(uv_1+vv_2+wv_3)\,d\tau+a\zeta=\text{const}$$

Let us multiply (1.11) by γ_1, γ_2, $1+\zeta$, respectively, and add the results. We thus find another integral

$$(A\omega_1+g_1)\gamma_1+(A\omega_2+g_2)\gamma_2+C\xi+\eta+[C(\omega+\xi)+g+\eta]\zeta=\text{const} \tag{1.18}$$

If we multiply the equation (1.13) by γ_1, γ_2, $1+\zeta$, respectively, and itemize by parts, we shall then find a further integral

$$\gamma_1^2+\gamma_2^2+\zeta^2+2\zeta=0 \tag{1.19}$$

In the investigation of the stability of the unperturbed motion we shall consider the perturbed motions in the general form without restricting them in any way.

2. Certain conditions of stability. The mechanical system under consideration is a conservative one, therefore it cannot be asymptotically stable. Indeed, it follows from Chetaev's theorem [10] that to every characteristic number different from zero there corresponds a negative characteristic number; the perturbed motion of the system will thus be unstable. Hence, the unperturbed motion of the system can be stable only in the case when all the characteristic numbers of the system are zero, i.e. only in the case that is critical in the sense of Liapunov, when the first approximation does not suffice to draw conclusions about the stability of motion.

In our problem, however, the integrals (1.15) to (1.19), as well as their linear combinations are not of definite sign relative to the variations of the variables (1.9). In order to bypass this difficulty, we make use of the following inequality established by Liapunov [6], which in our notation has the form:

$$g_1^2 + g_2^2 + g_3^2 \leqslant 2T_2 S \tag{2.1}$$

Here S denotes a quantity which is proportional to the largest of the principal moments of inertia (for the point O) of the liquid at any arbitrary instant of time.

Taking into account the notation (1.10), the inequality (2.1) for the perturbed motion of the liquid can be rewritten in the following form:

$$g_1^2 + g_2^2 + (g + \eta)^2 \leqslant 2T_2 S \tag{2.2}$$

where T_2 stands for the kinetic energy of the liquid in the perturbed motion, while g_1, g_2 and η are determined by the formulas (1.14).

Let us introduce into our consideration the function

$$H_1 = \tfrac{1}{2}[A\omega_1^2 + A\omega_2^2 + C(\omega + \xi)^2] + \tfrac{1}{2S}[g_1^2 + g_2^2 + (g + \eta)^2] + \\ + \tfrac{1}{2}M_1(v_1^2 + v_2^2 + v_3^2) + a(1 + \zeta) \tag{2.3}$$

Turning our attention to the inequality (2.2), we can convince ourselves that, for the consideration of the perturbed motion of the system, we have the following inequality:

$$H_1 \leqslant H, \qquad H = T + U = h \tag{2.4}$$

Here H denotes the total mechanical energy of the system during its perturbed motion.

Making use of the function H_1 and of the integrals (1.15), (1.18) and (1.19), we construct, by means of a linear combination of these functions, a new function V that is of definite sign.

In view of the inequality (2.4), the function V will be bounded from above and the conditions of its positive definiteness yield (in accordance with Liapunov's theorem on stability) sufficient conditions for the stability of the rotational motions of the solid body and of the liquid contained in the cavity. In this manner the function V serves to solve the problem on the stability within that bounded region of the space of variables ω_1, ω_2, ξ, g_1, g_2, η, γ_1, γ_2, ζ, v_1, v_2, v_3, where the equation $V = 0$ yield a system of closed bounded surfaces.

Let us now construct the function V.

From the integrals (1.18) and (1.19) we find

$$\mu = \mathrm{const} - (A\omega_1 + g_1)\gamma_1 - (A\omega_2 + g_2)\gamma_2 - C\xi - C(\omega + \xi)\zeta - g\zeta - \eta\zeta \tag{2.5}$$

$$\zeta = -\tfrac{1}{2}(\gamma_1^2 + \gamma_2^2 + \zeta^2) \tag{2.6}$$

Substituting for η and ζ in the expression (2.3) their values given by the formulas (2.5) and (2.6), dropping the unessential constants, and adding the terms $\xi^2 C(C-A)/2A$ and $C(g/S - \omega)\xi$, we obtain the function

$$\begin{aligned} V = \tfrac{1}{2}A(\omega_1^2 + \omega_2^2) + \frac{1}{2S}(g_1^2 + g_2^2 + \eta^2) - \\ - \frac{g}{S}[(A\omega_1 + g_1)\gamma_1 + (A\omega_2 + g_2)\gamma_2 + C\xi\zeta + \eta\zeta] + \\ + \frac{C^2}{2A}\xi^2 + \tfrac{1}{2}\left(C\omega\frac{g}{S} + \frac{g^2}{S} - a\right)(\gamma_1^2 + \gamma_2^2 + \zeta^2) + \tfrac{1}{2}M_1(v_1^2 + v_2^2 + v_3^2) \end{aligned} \tag{2.7}$$

which can be rewritten in the form

$$V = W_1 + W_2 + W_3 + \tfrac{1}{2}M_1(v_1^2 + v_2^2 + v_3^2)$$

Here we have introduced the following notations:

$$2W_1(\omega_1, g_1, \gamma_1) = A\omega_1^2 - 2\frac{g}{S}(A\omega_1 + g_1)\gamma_1 + \frac{1}{S}g_1^2 + \left(C\omega\frac{g}{S} + \frac{g^2}{S} - a\right)\gamma_1^2 \tag{2.8}$$

$$2W_3(\xi, \eta, \zeta) = \frac{C^2}{A}\xi^2 - 2\frac{g}{S}(C\xi + \eta)\zeta + \frac{1}{S}\eta^2 + \left(C\omega\frac{g}{S} + \frac{g^2}{S} - a\right)\zeta^2$$

and the function $W_2(\omega_2, g_2, \gamma_2)$ is similar to the function $W_1(\omega_1, g_1, \gamma_1)$.

The discriminants of the quadratic forms W_1 and W_3 can be written explicitly as

$$\begin{vmatrix} A & 0 & -\frac{A}{S}g \\ 0 & \frac{1}{S} & -\frac{g}{S} \\ -\frac{A}{S}g & -\frac{g}{S} & (C\omega + g)\frac{g}{S} - a \end{vmatrix} \qquad \begin{vmatrix} \frac{C^2}{A} & 0 & -\frac{C}{S}g \\ 0 & \frac{1}{S} & -\frac{g}{S} \\ -\frac{C}{S}g & -\frac{g}{S} & (C\omega + g)\frac{g}{S} - a \end{vmatrix}$$

In accordance with the criterion of Sylvester a quadratic form is positive-definite if and only if all of the principal diagonal minors of its discriminant are positive. In consequence of this we obtain the following condition for the positive definiteness of the functions W_1, W_2 and W_3:

$$\left(C\omega - \frac{A}{S}\, g\right) g - aS > 0. \tag{2.9}$$

Thus, if the condition (2.9) is fulfilled then the function V is positive-definite relative to the variables ω_1, ω_2, ξ, g_1, g_2, η, γ_1, γ_2, ζ, v_1, v_2, v_3; the function V is bounded from above. Hence, in accordance with Liapunov's theorem, the relation (2.9) is a sufficient condition for the stability of the rotational motion of a liquid-filled solid body in terms of the quantities (1.9).

The positive-definite function V can also be constructed in a different way [10]. Let us multiply the function (2.3) by $C\omega$, the integral (1.19) by $2a$, and add the results. After this we replace ζ by its value as given in (2.6) and add the terms

$$C\,(2a - C\omega^2)\,\xi, \qquad \frac{C^2\omega\,(C - A)}{2A}\,\xi^2. \tag{2.10}$$

We thus obtain, up to within some constants, the following expression for V

$$V = \frac{AC\omega}{2}(\omega_1^2 + \omega_2^2) + \frac{C^3\omega}{2A}\xi^2 + \frac{C\omega}{2S}(g_1^2 + g_2^2 + \eta^2) -$$
$$- 2a\,[(A\omega_1 + g_1)\gamma_1 + (A\omega_2 + g_2)\gamma_2 + (C\xi + \eta)\zeta] +$$
$$+ \left(\frac{C\omega}{2} + g\right) a\,(\gamma_1^2 + \gamma_2^2 + \zeta^2) + \tfrac{1}{2}\, C\omega M_1 (v_1^2 + v_2^2 + v_3^2) + \left(\frac{C\omega}{S}\, g - 2a\right)\eta$$

or

$$V = W_1 + W_2 + W_3 + \frac{C\omega}{2} M_1 (v_1^2 + v_2^2 + v_3^2) + \left(\frac{C\omega}{S}\, g - 2a\right)\eta \tag{2.11}$$

where we have made use of the following notation

$$W_1(\omega_1,\ g_1,\ \gamma_1) = \frac{AC\omega}{2}\,\omega_1^2 - 2a\,(A\omega_1 + g_1)\gamma_1 + \frac{C\omega}{2S}\, g_1^2 + \left(\frac{C\omega}{2} + g\right) a\gamma_1^2$$

$$W_3(\xi,\ \eta,\ \zeta) = \frac{C^3\omega}{2A}\,\xi^2 - 2a\,(C\xi + \eta)\zeta + \frac{C\omega}{2S}\,\eta^2 + \left(\frac{C\omega}{2} + g\right) a\zeta^2$$

The function $W_2(\omega_2,\ g_2, \gamma_2)$ is analogous to $W_1(\omega_1,\ g_1,\ \gamma_1)$. Writing down explicitly the discriminants of the quadratic forms W_1 and W_3

$$\begin{vmatrix} \frac{AC\omega}{2} & 0 & -Aa \\ 0 & \frac{C\omega}{2S} & -a \\ -Aa & -a & \left(\frac{C\omega}{2} + g\right) a \end{vmatrix} \qquad \begin{vmatrix} \frac{C^3\omega}{2A} & 0 & -Ca \\ 0 & \frac{C\omega}{2S} & -a \\ -Ca & -a & \left(\frac{C\omega}{2} + g\right) a \end{vmatrix}$$

in accordance with Sylvester's criterion we obtain the following conditions for the positive definiteness of the forms W_1, W_2, W_3

$$C^2\omega^2 + 2C\omega g - 4(A+S)a > 0 \tag{2.12}$$

If during the entire course of motion

$$\left(\frac{C\omega}{S} g - 2a\right)\eta \geqslant 0 \tag{2.13}$$

then, under condition (2.12), the function V will be positive-definite and bounded from above in view of the inequality (2.4). By Liapunov's theorem, the conditions (2.12) and (2.13) will then be sufficient conditions, relative to the quantities (1.9), of a hollow solid body containing a liquid.

It should be noted that in the absence of the liquid within the hollow of the body (when one has to suppose that $g = 0$, $S = 0$, $g_1 = g_2 = \eta = 0$), the condition (2.12) goes over into the well known condition of Maievskii for the stability of projectiles

$$C^2\omega^2 - 4Aa > 0$$

As an example let us consider the case of the potential of the motion of a liquid completely filling a cavity shaped like a body of revolution. As is known [1], the uniform rotation of the body around the Oz-axis of the cavity does not give rise to any motion of an ideal liquid that is at rest, and $g = 0$. In the perturbed motion, the velocities of the particles of the liquid are determined by the potential $\phi(x, y, z, t)$ of the velocities, where

$$\varphi = \psi_1(x, y, z)\,\omega_1 + \psi_2(x, y, z)\,\omega_2$$

and one can easily see that

$$\eta = \rho \int_\sigma \varphi (xm - yn)\, d\sigma = 0$$

because along the walls of the cavity σ the following condition holds

$$xm - yn = 0$$

where m and n denote the direction cosines of the exterior normal to the surface σ. Hence, in this case the condition (2.13) is satisfied while condition (2.12) takes on the following form

$$C^2\omega^2 - 4(A+S)a > 0 \tag{2.14}$$

When this condition is fulfilled, the rotational motion of a solid body with a liquid content will be stable.

We note that a stability condition of the type (2.14), valid for all bodies containing a liquid with vortexless motion, was first obtained by

Chetaev [2] when he was considering the stability of an equivalent solid body; in that case the quantity S represented the equatorial moment of inertia of the equivalent solid body.

In conclusion we emphasize once more that the stability of the rotational motion of a solid body with liquid content under the action of a tilting moment is attained by means of gyroscopic stabilization, and that the latter, as was shown by Kelvin, cannot occur when the system is acted upon by a dissipative force with a loss of energy under arbitrary displacements of the system. Since in real situations there always exist small dissipative forces, the gyroscopic stability will always be destroyed. In view of this, Kelvin suggested that a distinction should be made between "temporary" stability, which can be achieved with gyroscopic stabilization, and "secular" (or permanent) stability which exists under the action of potential forces only. It is obvious that the stability within our system has the character of the "temporary" stability, and that our system is unstable in the "secular" sense. Chetaev [10] succeeded in proving a theorem on the instability of the motion of a solid body by taking into account dissipative forces. It seems that this theorem is also valid for the rotational motion of a solid with a cavity containing a liquid.

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